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Performance evaluation of a single queue under multi-user TCP/IP connections version 2

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Abstract: We study the performance of several TCP connections through the bottleneck of a slow network accessed via a single queue with high but finite capacity. Using mean-field asymptotic methology, we establish some asymptotical results about queue length distribution and window size distribution when the number of user increases proportionally to buffer capacity. We show that the difference between the actual queue length and its maximal capacity tends to be exponentially distributed. We give a precise determination of the window size asymptotic distribution and we prove that under our model the small window size has log-normal distribution.

Key-words: Internet, TCP, asymptotics, mean-field approximation, fairness

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Évaluation de performance d'une file d'attente unique soumise à des connexions TCP multiples, version 2

Résumé : Nous étudions les performances de plusieurs connexions TCP soumises au goulot d'étranglement d'un réseau lent desservi par une file d'attente de grande capacité. En utilisant la méthode asymptotique du champ moyen nous établissons des résultats asymptotiques sur la distribution de la longueur de la file d'attente et des tailles de fenêtre quand le nombre d'utilisateurs croît proportionnellement 'a la capacité de la file d'attente. Nous montrons que la quantité de places libres dans le buffer d'attente suit une loi exponentielle. Nous évaluons précisément la distribution de la longueur des fenêtres de retransmissions. Nous prouvons que cette distribution est log-normale.

Mots-clés : Internet, TCP, évaluation de performances, comportement asymptotique, approximation par le champ moyen, équité.

1 Introduction

The protocol TCP [1] is widely used in the Internet (Web, FTP). The proportion of connections made on this protocol is overwhelming (close to 99.9%). The protocol TCP is an end-to-end flow control protocol based on dynamic transmission windows. The protocol does not make any assumption on the underlying network and on how heterogeneous networks are connected. The reason of the success of TCP is mainly based on its high dynamic that make it able to adapt itself to any kind of network capacity from few bauds to several gigabit per second.

The protocol TCP has received special attention since its formulation as an internet standard. In particular its performance has been the research focuse of several researcher. The paper [3] is a pioneer paper where the performance of TCP is analyzed in the specific case of a single TCP connection on a single router connected to an infinite capacity channel with a fixed independent error rate p. More recently Baccelli et al. [2] have analyzed a single TCP connection through a sequence of finite capacity routers. The analysis is interesting because it relyes on an explicit formulation of the problem in (max,plus) algebra. The problem of several connections even a single router has been only simulated with the notable exception of [4], but limited to very specific phases of the protocol.

The aim of this paper is to investigate via analytic methods the multi-connection cases where N TCP connections coexist on the same finite capacity router connected to a finite capacity network. We denote B the finite capacity of the router, and $T = B/N$. Extending the already difficult single connection case to the multi-connection case is absolutely out of reach of the present performance evaluation toolbox. However our analysis is made possible because we consider the asymptotic case where N is large and the analysis is much simplified. Indeed the calculation of the steady states turns out to be much simpler compared to the single connection case.

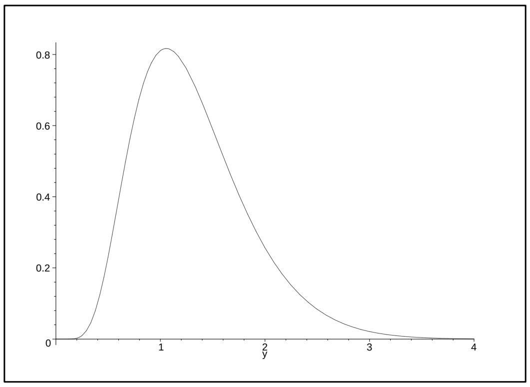
Practically we investigate the case where N users access N server under TCP/IP and the bottleneck is a router with a finite buffer and a slow network interface. The servers are not necessarily different, as well as the users, but we assume that N connections are active. The network is divided into two areas:

- a local loop with relatively low speed (telephone line, cable TV, ADSL)
- a backbone with high throughput (ATM, DWDM)

We assume that the users are located on the local loop and the servers are located on the fast backbone. We assume that there is a router at the border of the local loop and the fast backbone (head-end, etc).

We will consider that every user is downloading a file of infinite size and we are interested into analyzing the steady state of every connection. We also assume that the round trip delay between server and user is large.

This paper is version 2 of [8]. The paper is divided into the following sections. A first section is devoted to a short presentation of TCP. A second section describes of the models. A third section presents the analysis of a simplified model and the result are used to the analysis of a more realistic model. A last section is devoted to numerical example with interpretations about traffic fairness. In particular it will be shown that the remaining space in the queue tends to be exponentially distributed when $N \to \infty$. When T and N tend both to infinity, the window size distribution rescaled by $T^{-1/2}$ tends to a continuous distribution $g(y)$ given by figure 1. In [9] the same limiting function has been extracted in the more general approach of TCP control with side control traffic as in the RED protocol.



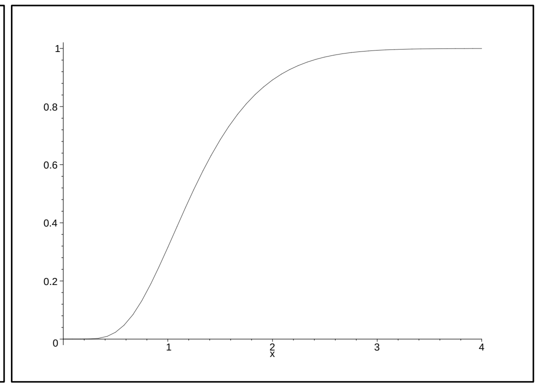


Figure 1: Limiting function $g(x)$ and its primitive for window size distribution.

In particular, function $\log g(y) \approx -\log 2 \log^2 y$ when $y \to 0$, therefore small window size distribution is log-normal as illustrated in figure 2. Since the window size is proportional to the instantaneous throughput of the connection, we have proven that the throughput is log-normally distributed.

2 The TCP connection protocol and its model

2.1 TCP overview

The connections are done under a window protocol like TCP/IP. Packets are transmitted in order and must be acknowledged by the end-user. A packet is loss when the acknowledgement

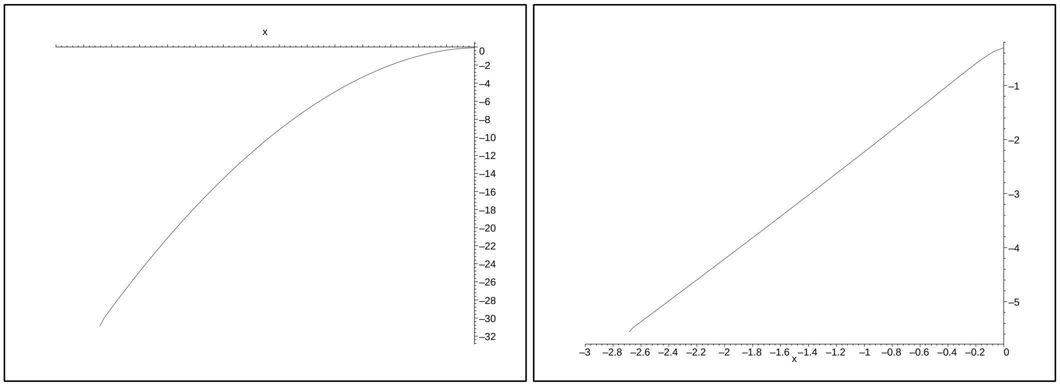


Figure 2: Limiting function $\log_{10} g(10^x)$ and $-\sqrt{\log_{10} g(10^x)}$ as function of x.

does not arrive within the estimated round trip delay. A packet loss is considered as a congestion event.

In order to cope with the round trip delay, several packets are transmitted in advance without waiting for acknowledgement. The set of unacknowledged packets is called the window and its size varies in order to handle congestion.

- when no packet is lost in a window (successful window), the next window size is incremented of 1 unit.
- when a packet loss occur (failed window), packet retransmission starts from this packet but with a window size halved.

There are several feature that add to the previous one that allows TCP to be more reactive on networks event. Among them are self-clocking and slow start.

Self-clocking The server updates its window size on-line with each acknowledgement return. The server synchronizes the transmission of its new packets with the acknowledgement returns leading to a *self-clocking* of packet transmissions. Packets can be acknowledged in batch *via* appropriate tuning of parameter k.

Slow start There is also the *slow start* process where the window is doubled at each successful window as long the window size is below a pre-defined threshold. We will not discuss of this feature here, since we focus on steady state analysis.

2.2 The models

2.2.1 The continuous model with batch transmission

We consider that the router has a very large buffer but whatever its size the window protocol strives to almost fill it in steady state situation. The parameter of interest is the current available room in the buffer at time t called $R(t)$.

We assume that the buffer contains continuous data (fluid approximation), and $R(t)$ is a positive real number. The buffer is served at speed $\mu > 0$.

The current window size at server i is $W_i(t)$, which is also a real number. We consider the following simplified mode of operation:

- the server transmits all the packets within its window at the same time;
- The backbone has an infinite speed so that all the packets of the same window arrives simultaneously in the buffer;
- the end-user acknowledges all the packets received from the same window in the same packet (aggregated acknowledgement).

When server i transmits its window, we assume

1. If the current window $W_i(t) < R(t)$, then $R(t)$ becomes $R(t) - W_i(t)$ and $W_i(t)$ becomes $W_i(t) + 1$;
2. else, if $W_i(t) \geq R(t)$, then $R(t)$ becomes 0, and $W_i(t)$ becomes $W_i(t)/2$.

2.2.2 The Discrete model

In this model we suppose that the buffer length $R(t)$ and the window length $W_i(t)$ are discrete, $r = 1, 2, ...,$ $w_i = 1, 2, ..N$ are integers.

It is supposed that the length of the window is increased by 1, or $w_i \to \lceil w_i/2 \rceil$ at any step the window is addressed (here we have to specify that when $w = 2i - 1$ it becomes i); the length of free buffer can increase by 1 or can be made 0, following the model proposed above.

We will mainly focus our presentation on the continuous model, but we will outline some sketches from the discrete model as often as possible.

2.2.3 The exponential round trip delay

We make a further approximation by assuming that windows arrive on buffer according to a Poisson process of rate $\lambda > 0$. Within this model the round trip delays per server is i.i.d. and is Poisson with mean N/λ. Windows arrive on buffer according to a Poisson process of rate λ. A further level of abstraction in the model consists into monitoring the arrival point as Poisson events and at each Poisson event to randomly select the transmitter server i, uniformly in $(1, N)$. Within this model there is no need to keep track the exact matching between server and window size. It suffices to keep track of the repartition function $W(y, t)$:

$$W(y,t) = \frac{\text{number of servers at time } t \text{ with window size} \geq y}{N} \tag{1}$$

With $W(0, t) = 1$.

The exponential round trip model is convenient for a first analysis but it is highly non realistic. First, it varies in too large proportion: a difference between two consecutive round trip delay will be interpreted as a packet loss by the server and cause a window halving. Second the propagation delay contains the buffer delay experienced by the last packet of the window, *i.e.* exactly $\frac{B-R(t)}{\mu}$ which is not expected to have an exponential distribution. In fact it will be proven that the buffer delay will be close to $\frac{B}{\mu}$.

2.2.4 A realistic model with fixed delay plus random processing time

In this model we assume that the propagation delay has two components:

- A fixed buffer delay $NT = \frac{B}{\mu}$;
- A random exponential delay of mean $\frac{N}{\lambda}$, assumed to be much smaller than NT.

We can see the factor N as the nominal buffer capacity per server. For the convenience of the presentation we will call the small exponential delay the *processing time*, but it can be just a component of the propagation delay, for example some buffer time in the high speed part of the network.

In this model, the server are either (i) transmitting, (ii) in fixed propagation delay or (iii) in random processing delay. We keep the repartition function $W(y, t)$ but with a different meaning:

$$W(y,t) = \frac{\text{number of servers in processing at time } t \text{ with window size} \geq y}{N} \tag{2}$$

In this case $W(0, t) < 1$. Indeed, quantity $W(0, t)$ is equal to the proportion of server in processing at time t. The average value of $W(0, t)$ over t is equal to $\frac{1}{\lambda T+1}$.

This very model is much realistic than the previous one and can be treated as well. But we will first handle the first unrealistic model which will give the foundations of our framework.

3 Main results

3.1 Notations and system description

We denote $R(x,t) = P(R(t) > x)$ and $w(y,t) = \frac{-\partial}{\partial y} W(y,t)$, *i.e* the density of window size distribution.In other words, $w(y,t) = \frac{1}{N} \sum_{i=1}^{i=N} \delta(y - W_i(t))$ where $\delta()$ is the Dirac function.

In the model with processing time, this is equal to the window size density of server in processing state. We shall immediately outline two important points:

1. The quantity $R(x,t)$ addresses a probability distribution.
2. The quantity $W(y,t)$ addresses a state function of the system and *a priori* is not a probability distribution

Therefore quantities $R(x,t)$ and $W(y,t)$ are not sufficient to describe the probabilistic behavior of the system. The complete probabilistic description of the system should be given by function $\rho(x,f,t) = P(R(t) > x, W(y,t) = f(y))$, where $f(.)$ is a positive function.

3.2 Results

In this section we mainly deal with the exponential delay model. We will also show preliminary results for the fixed delay plus exponential processing time. We present the results equations that describe the performance of this model and justify the asymptotic approach.

3.2.1 Fixed window distribution and fixed $R(t)$ distribution models

We consider the side system where the distribution of W is fixed and does not change with the $R(t)$. This is not the real system since $R(t)$ and $W(t)$ are actually dependent. Therefore we denote this fake system $\tilde{R}(t)$ and $\tilde{w}$ denote the fixed window distribution. We call this model the Fixed Window Distribution (FWD) model.

Lemma 1 *In the FWD model the functional equation of* $\tilde{R}(x,t) = P(\tilde{R}(t) > x)$ *is*

$$\frac{\partial \tilde{R}(x,t)}{dt} = \mu \frac{\partial \tilde{R}(x,t)}{\partial x} - \lambda \tilde{R}(x,t) + +\lambda \int_{k=0}^{\infty} \tilde{R}(x+y,t) \tilde{w}(y,t) dt, \tag{3}$$

Proof: In absence of window transmission, quantity $R(t)$ increases at rate μ. Windows are transmitted at rate λ. When a window of size y (which occurs with differential probability $\tilde{w}(y)$) is transmitted between t and $t+dt$ (dt infinitely small) two cases are possible:

- $R(t) > y$, in this case $R(t+dt) = R(t) - y$.
- $R(t) \leq y$, in this case $R(t+dt) = 0$

Translating these events into stochastic equations leads to equation (3). ■

We assume now that there is $a > 0$ such that $E[e^{aW}] < \infty$.

Lemma 2 *In FWD model let assume a that $E[\tilde{W}] > \mu/\lambda$. The steady state distribution of $\tilde{R}(t)$ is exponential with parameter $\tilde{a} > 0$: $\lim_{t\to\infty} P(\tilde{R}(t) > x) = \exp(-\tilde{a}x)$. Parameter $\tilde{a}$ satisfies:*

$$\lambda(1 - \mathrm{E}[e^{-\tilde{a}\tilde{W}}]) = \mu\tilde{a} \tag{4}$$

and the convergence rate to steady state is at least $\max_\omega\{\mu\omega - \lambda(\mathrm{E}[e^{\omega\tilde{W}}] - 1)\}$

Proof: Let consider an initial distribution such that $\mathrm{E}[e^{a\tilde{R}(0)}] < A$ for some $A > 0$.

We denote by $R^*(\omega)$ the Laplace transform $\mathrm{E}[e^{-\omega\tilde{R}(t)}]$. From equation (3) we get by Laplace:

$$\frac{\partial}{\partial t}R^*(\omega,t) = -\mu - \mu\omega R^*(\omega,t) + \lambda R^*(\omega,t)(\mathrm{E}[e^{\omega W}] - 1)\ . \tag{5}$$

Therefore the stationary distribution is Poisson of rate $a > 0$ such that

$$\lambda(1 - \mathrm{E}[e^{-aW}]) = \mu a \tag{6}$$

Moreover if two initial distributions $\tilde{R}_1(x,0)$ and $\tilde{R}_2(x,0)$ satisfies $\mathrm{E}[e^{aR(0)}] < \infty$. Therefore their transient distribution, characterized by their Laplace transform $R_1^*(\omega,t)$ and $R_2^*(\omega,t)$ tend to converge.

Namely it comes from 5 that

$$R_1^*(\omega,t) - R_2^*(\omega,t) = \exp(-(\mu\omega - \lambda(\mathrm{E}[e^{\omega W}] - 1))t)(R_1^*(\omega,0) - R_2^*(\omega,0)) \tag{7}$$

and the convergence is exponential of rate $\max_\omega\{\mu\omega - \lambda(\mathrm{E}[e^{\omega W}] - 1)\} > 0$. ■

3.2.2 The Independent $R(t)$ Distribution models

We can also define the Independent $R(t)$ Distribution (IRD) model where the distribution of $R(t)$ is time independent, *i.e* $R(t)$ is independent of $R(t')$ as soon as $t \neq t'$. We denote by $\tilde{R}(x,t)$ the distribution of $R(t)$. Let i be a fixed integer in $[1,N)$, for example $i = 1$.

Let $w_i(y,t)$ be the distribution of $W_i(t)$, the window of server i. We denote $w_i^N(y,t) = w_i(y,Nt)$.

Lemma 3 *In the IRD model the evolution equation of $w_i^N(y,t)$ is*

$$\frac{\partial}{\partial t}w_i^N(y,t) = \lambda\tilde{R}(y-1,t)w_i^N(y-1,t) + 2\lambda(1-\tilde{R}(2y,t))w_i^N(2y,t) - \lambda w_i^N(y) \tag{8}$$

Remark The independence hypothesis of $R(t)$ with $R(t')$ as soon as $t \neq t'$ can be relaxed by the weaker hypothesis that $R(t)$ is independent at each transmission times of server i.

Proof: After time re-scaling by factor N, window transmission of server i occur at rate λ. If at time t, server i transmit and its window size is y, then either

- $R(t) < y$, and $W_i(t+dt) = y/2$
- $R(t) \geq y$ and $W_i(t+dt) = y+1$.

These alternatives translated into stochastic equation (8).■

3.2.3 Main theorems

The kernel of our result is in the following theorem.

Theorem 1 *In the real system, when $N \to \infty$, let assume that $\lim_{N\to\infty} W(y,0) = \tilde{W}(y,0)$ and $\lambda\int\tilde{W}(y,0)dy > \mu$. In this case $\lim_{N\to\infty} W(y,Nt) = \int_0^y w(x,t)dx$ such that:*

$$\frac{\partial}{\partial t}w(y,t) = \lambda e^{-a(t)(y-1)}w(y-1,t) + 2\lambda(1-e^{-2a(t)y})w(2y,t) - \lambda w(y) \tag{9}$$

such that $a(t)$ is the non-negative solution of

$$\lambda(\int(1-e^{-a(t)y)})w(y,t)dy = \mu a(t). \tag{10}$$

Furthermore $R(t)$ is exponential of parameter $a(t/N)$.

Proof: The proof of the theorem is given indetail in the appendix. It proceeds in three steps:

(i) $\partial W(y,t)/\partial t = O(\lambda/N)$

(ii) $R(t)$ is exponential of rate $a(t/N)$

(iii) $\partial W(y,t)/\partial y$ tends to be equal to $\frac{1}{N}\sum_i w_i(y,t)$.

Point (i) is given by the fact that during any time interval of length Δt an average of $\lambda\Delta t$ servers transmit leading to a modification of order $O(\lambda\Delta t/N)$ of repartition function $W(y,t)$.

Point (ii) comes from the fact that $R(t)$ has time to converge to the exponential steady state of parameter $a(t/N)$ as in FWD model before $W(y,t)$ has enough time to significantly change, when N increases. For example, given $a(t/N)$ given by (15) at time t, the random variable $R(x,t)$ has time to converge to $\exp(-a(t/N)x)$ during the interval $[t, t+\Delta t\sqrt{N}]$ during which $W(y,t)$ remain inchanged at $\pm O(\lambda\Delta t/\sqrt{N})$.

Point (iii) comes first from the fact that for any given fixed server, the distribution of $R(t)$ is independent and exponential of parameter $a(t/N)$ at each transmission time of the server when $N \to \infty$. Therefore the IRD model applies. Second a straightforward application of the mean-field approximation to the system leads to $W(y,t) = 1 - \frac{1}{N}\sum_i \int_0^y w_i(x,t)dx + O(\frac{1}{\sqrt{N}})$.
■

Corollary 1 *When $N \to \infty$ the steady state of window size is the function $w(y)$ which satisfies the equation:*

$$w(y) = e^{-a(y-1)}w(y-1) + 2(1-e^{-2ay})w(2y) \ . \tag{11}$$

and

$$\int w(y)dy = 1 \tag{12}$$

where a is the non-negative solution of

$$\frac{\int(1-e^{-ay})w(y)dy}{a} = \frac{\mu}{\lambda} \tag{13}$$

It is clear that the best way to solve the above equations is to use a as a parameter and then to express λ/μ as a function of a.

The realistic model where the round trip delay of each server is a constant delay NT plus a processing time before transmission exponentially distributed with parameter λ/N (therefore an average round trip delay equal to $NT + N/\lambda$) can be expressed by a modification of theorem 1. In the following theorem $W(y,t)$ now denotes the distribution of window size on the server in processing time. In particular $W(0,t) < 1$ and its average value is $\frac{1}{\lambda T+1}$.

Theorem 2 *In the fixed delay plus exponential processing time model, when $N \to \infty$, let assume that for $t \in [0,T]$ $\lim_{N\to\infty} W(y,t) = \tilde{W}(y,t)$ and $\lambda\int\tilde{W}(y,t)dy > \mu$. In this case $\lim_{N\to\infty} W(y,Nt) = \int_0^y w(x,t)dx$ such that:*

$$\frac{\partial}{\partial t}w(y,t) = \lambda e^{-a(t-T)(y-1)}w(y-1,t-T) + 2\lambda(1-e^{-2a(t)y})w(2y,t-T) - \lambda w(y) \tag{14}$$

such that $a(t)$ is the non-negative solution of

$$\lambda(\int(1-e^{-a(t)y)})w(y,t)dy = \mu a(t). \tag{15}$$

Furthermore $R(t)$ is exponential of parameter $a(t/N)$.

Proof: We don't have yet the complete proof of the theorem (which is therefore formally a conjecture). The proof however proceeds the same way as with theorem 1 but is significantly complicated by the retarded mechanism.

Corollary 2 *When $N \to \infty$ the steady state of window size is the function $(\lambda T + 1)w(y)$ which satisfies the equation (11) and (13) but with*

$$\int w(y)dy = \frac{1}{\lambda T + 1} \tag{16}$$

4 Large round trip delay

4.1 Window size stationnary distribution

Our aim is to exploit the results obtained in previous section in the case where the round trip delay is large (*i.e* when T or $1/\lambda$ is large). In this case the quantity a that characterizes the Poisson parameter of the qunatity $R(t)$ (the buffer free space) is small.

We are interested into the unconditional distribution of the window size. It is interesting to consider case when $1 \gg a$.

Theorem 3 *When $a \to 0$ the window size distribution $w(y)$ satisfies:*

$$\lim w(y/\sqrt{a})/\sqrt{a} = g(y) \tag{17}$$

where

$$g(y) = \sqrt{\frac{2}{\pi}} \prod_{k \geq 0} (1 - 4^{-k} 2^{-1})^{-1} \sum_{n \geq 0} a_n \exp(-4^n y^2/2) \ , \tag{18}$$

with a_n satisfying the Taylor identity: $\sum_{n \geq n} a_n x^n = \prod_{k \geq 0} (1 - 4^{-k} x)$.

And

$$\sqrt{a} = (1 + O(\sqrt{a})) \frac{\lambda}{\mu} g^*(2) \tag{19}$$

with $g^*(2) = \sqrt{\frac{2}{\pi}} \prod_{k \geq 0} \frac{1 - 4^{-k-1}}{1 - 4^{-k} 2^{-1}} \approx 1.309833$.

Let's make the change of variable can consider $w(y) = \sqrt{a} g(y\sqrt{a})$. Equation (11) rewrites

$$g(y) = e^{-y\sqrt{a} + a} g(y - \sqrt{a}) + 2(1 - e^{-2y\sqrt{a}}) g(2y) \tag{20}$$

when $a \to 0$ the equation expanded to first order in $\sqrt{a}$ becomes

$$(1 - y\sqrt{a})(g(y) - \sqrt{a}g'(y)) + 4y\sqrt{a}g(2y) = g(y) \tag{21}$$

where $g'(y)$ is the first derivative of function $g(y)$ at point y.

Simplifying we obtain the differential equation:

$$yg(y) + g'(y) = 4yg(2y) \tag{22}$$

This equation is easy to solve via Mellin transform $g^*(s) = \int_0^\infty g(y)y^{s-1}dy$:

$$g^*(s+1) + (s-1)g^*(s-1) = 2^{1-s}g^*(s+1) \tag{23}$$

By fixing $g^*(s) = v(s)2^{s/2}\Gamma(s/2)$ we get

$$v(s) = v(s+2)(1 - 2^{-s}) \ . \tag{24}$$

The above formula is easy to solve with $v(s) = \alpha \prod_{k\geq 0}(1 - 2^{-s-2k})$. Therefore

$$g^*(s) = \alpha 2^{s/2}\Gamma(s/2) \prod_{k\geq 0}(1 - 4^{-k}2^{-s}) \tag{25}$$

The value of α is extracted from the identities $g^*(1) = \int w(y)dy = 1$, and it comes

$$g^*(s) = \sqrt{\frac{1}{2\pi}} 2^{s/2}\Gamma(s/2) \prod_{k\geq 0} \frac{1 - 4^{-k}2^{-s}}{1 - 4^{-k}2^{-1}} \tag{26}$$

Notice that function $g^*(s)$ has no singularity since poles of $\Gamma(s/2)$ at $s = -2k$, k natural integer, are exactly cancealed by being root of $1 - 4^{-k}2^{-s}$. Therefore the function $g(y)$ will have all its derivatives at zero at $y = 0$.

It comes that

$$\mathrm{E}[W] = \frac{g^*(2)}{\sqrt{a}} = \frac{2}{\sqrt{2\pi a}} \prod_{k\geq 0} \frac{1 - 4^{-k-1}}{1 - 4^{-k}2^{-1}} \tag{27}$$

and (13) becomes to

$$\frac{\mu}{\lambda} = \frac{1 - \int_0^\infty e^{-ay}w(y)dy}{a} \approx \mathrm{E}[W] \tag{28}$$

which leads $\lambda = \mu\sqrt{2\pi a} \prod_{k\geq 1} \frac{1-4^{-k}2^{-1}}{1-4^{-k-1}}$.

By reverse Mellin it comes that

$$g(y) = \sqrt{\frac{2}{\pi}} \prod_{k\geq 0}(1 - 4^{-k}2^{-1})^{-1} \sum_{n\geq 0} a_n \exp(-4^n y^2/2) \ , \tag{29}$$

with a_n satisfying the Taylor identity: $\sum_{n\geq n} a_n x^n = \prod_{k\geq 0}(1 - 4^{-k}x)$.■

Corollary 3 *When $a \to 0$ the average window size is*

$$\mathrm{E}[W] = \frac{2}{\sqrt{2\pi a}} \prod_{k\geq 0} \frac{1-4^{-k-1}}{1-4^{-k}2^{-1}} \tag{30}$$

The average packet retransmission $\mathrm{E}[W]\frac{\lambda}{\mu} - 1 \to 0$, and the average number of dropped packet per window $\frac{1}{2}(\mathrm{E}[W^2] \times a)$ tends to 1.

In the case of the fixed delay plus exponential processing time model we have the following theorem.

Theorem 4 *when $T + 1/\lambda \to \infty$,*

$$\lim(\lambda T + 1) w(y/\sqrt{a})/\sqrt{a} = g(y) \tag{31}$$

and

$$\sqrt{a} = (1 + O(\sqrt{a}))\frac{\lambda}{\lambda T + 1}\frac{g^*(2)}{\mu} \tag{32}$$

4.2 log-normal distribution of small windows

The aim of this section is to show that the distribution is *log-normal* for small windows, namely that $\log \Pr\{W < x\} = -\Theta(\log^2 x)$.

Theorem 5 *In large propagation delay condition we have $-\log Pr\{W < x\} \sim (\log_2 x)^2$ when $x \to 0$.*

Proof: We know that $\Pr\{W = x\} \approx \sqrt{a} g(\sqrt{a} x)$. The point is to prove that $\log g(x) \sim -(\log_2 x)^2$. We have

$$g(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s} g^*(s) ds \tag{33}$$

for some c in the real definition domain of $g^*(s)$, *i.e.* the whole real axis. We can rewrite

$$g(x) = \frac{\alpha}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s} 2^{s/2}\Gamma(s/2) \prod_{k\geq 0}(1 - 2^{-s-2k}) \tag{34}$$

With α, a constant defined in section xx, which has no impact on the asymptotic expansion.

We will define $c = -\frac{1}{2} - k(x)$ where $k(x)$ is an integer function of x. We define $k(x) = \arg\min_k\{-\log x - k\log 2 - \frac{1}{2}\log k\}$. For $x \to 0$ the minimum tends to be smaller to 1 and $k(x) \sim \log_2 x$.

For k integer we have

$$g^*(s-2k) = \alpha \prod_{\ell \geq 0} (1 - 2^{-s+2k-2\ell}) 2^{s/2-k} \Gamma(\frac{s}{2} - k) \tag{35}$$

We first use

$$\begin{aligned} \prod_{\ell \geq 0} (1 - 2^{-s+2k-2\ell}) &= \prod_{\ell=1}^{\ell=k} (1 - 2^{-s+2\ell}) \prod_{\ell \geq 0} (1 - 2^{-s-2\ell}) \quad (36) \\ &= (1 + O(2^{-k})) 2^{-ks+(k+1)k} \\ &\times \prod_{\ell \geq 0} (1 - 2^{s-2\ell})(1 - 2^{-s-2\ell}) \quad (37) \end{aligned}$$

We second use

$$\begin{aligned} \Gamma(\frac{s}{2} - k)\Gamma(k) &= \sqrt{2\pi} \frac{(\frac{s}{2} - k)^{\frac{s}{2}-k-1/2} k^{k-1/2}}{(\frac{s}{2})^{(s-1)/2}} \Gamma(\frac{s}{2}) \quad (38) \\ &= (-1)^k k^{\frac{s}{2}-1} (-1)^{(s-1)/2} e^{s/2} \Gamma(\frac{s}{2})(1 + O(\frac{1}{k})) \quad (39) \end{aligned}$$

Therefore

$$\begin{aligned} g^*(s-2k) &= (1 + O(\frac{1}{k}))\alpha 2^{-ks+(k+1)k-(s-1)/2} s^{(s-1)/2} k^{\frac{s}{2}-1} \\ &\times (-1)^{(s-1)/2} e^{s/2} \Gamma(\frac{s}{2}) 2^{s/2-k} \rho(s) \quad (40) \end{aligned}$$

with $\rho(s) = \prod_{\ell \geq 0} (1 - 2^{s-2\ell})(1 - 2^{-s-2\ell})$.

And

$$\begin{aligned} x^{-s+2k(x)} g^*(s - 2k(x)) &= (1 + O(\frac{1}{k}))\alpha \exp(-\Delta(x)s)(-1)^{(s-1)/2} e^{s/2} \Gamma(\frac{s}{2})\rho(s) \\ &\times \exp(2k(x)\log x + k(x)^2 \log 2 - \log k(x)) \quad (41) \end{aligned}$$

and

$$g(x) = \alpha \exp(2k(x)\log x + k(x)^2 \log 2 - \log k(x)) G(\Delta(x)) \tag{42}$$

where $G(y) = \frac{1}{2i\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} y^{-s}(-1)^{(s-1)/2} e^{s/2} \Gamma(\frac{s}{2})\rho(s) ds$.

Since $\Delta(x)$ is bounded and consequently $G(\Delta(x))$ we have $\log g(x) = -\log^2 x \log 2 + O \log x)$ and the theorem is proven.■

5 Simulation results

5.1 Simplified TCP

We have simulated the simplified TCP with different number of connections and different buffer size and random delay. By simplified TCP we mean the approximated version of TCP we have analysed in the paper, *i.e.*:

- windows are transmitted and acknowledged by batches;
- window sizes are real numbers;
- connection never come again in slow start.

Each of the connections start at a random time after time $t = 0$. The connection starts in slow start mode. A connection leaves the slow start mode when it meets its first congestion. The connections never come back to slow start mode. The simulation runs are enough long so that every connection has left its slow start mode and that the process has attained its stationary state to be compared with analytical results. In all simulations of simplified TCP, the buffer service rate μ is 1 packet per time unit.

5.2 Real TCP

We have simulated the real TCP using the ns2 simulation tool of [5]. The used TCP version is Reno. The link from the buffer to the clients is 8 Mbps. The packet size is 1K octets. Each server has a private link to the buffer at 1 Gbps (see figure 3).

5.3 Histograms of TCP connections

5.3.1 Simplified TCP

Figure 4, 5 show the histogram of the buffer occupancy during a certain interval of time. Figure 4 shows the buffer occupancy when the number of connections $N = 100$ and the buffer size is $B = 100$. Figure 5 shows the buffer occupancy when the number of connections $N = 100$ and the buffer size is $B = 300$. In both figures, the average processing time is 10. We also added a waterline which indicates the average buffer occupancy computed with the asymptotical formula of the section about window size distribution. For figure 4, 5 the mean field parameter (*i.e* the average number of servers in processing time) is respectively 10 and 3.2.

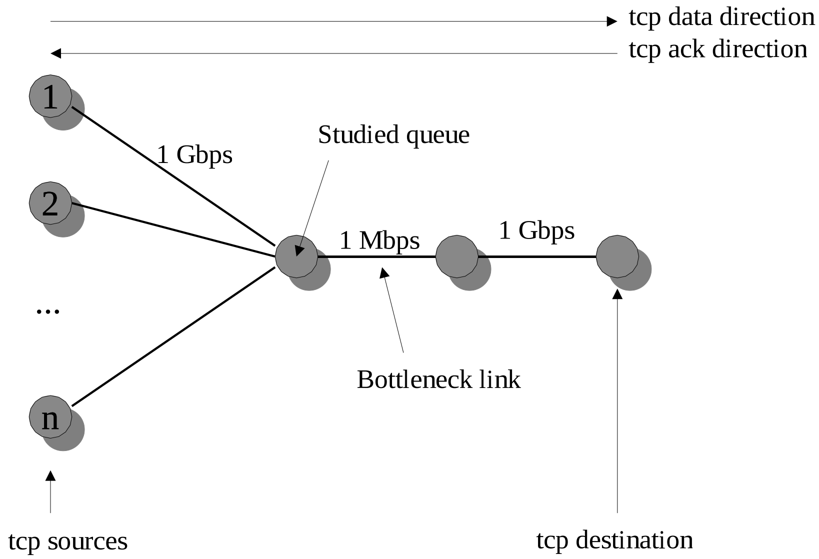


Figure 3: ns2 simulation diagram for TCP Reno

5.4 Buffer ocupancy distribution

5.4.1 Simplified TCP

Figure 6 displays the buffer occupancy distribution obtained via simulation. On each plot the scale is logarithmic and the straight line shows the theoretical exponential repartition function with the rate computed from limiting formula (32). In figure 6, the parameters are the same excepted that the average processing time is 100.

5.4.2 TCP Reno

Figures 7 displays the window distribution throughout the simulations. In each simulations, the buffer size has been sampled and the sample set fas been ranked from the smallest to the largest. The rank of size x divided by the number of samples gives the estimated probability $P(R(t) > x)$. Therefore a linear look in log scale outlines an exponential behavior. In every figure the number of samples is 35,000.

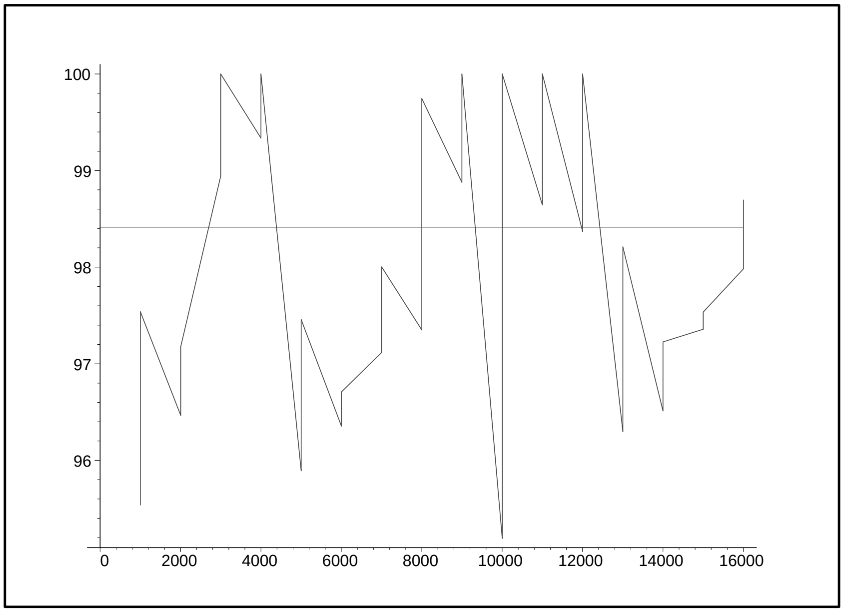


Figure 4: Histogram of buffer occupancy versus time for simplified TCP, for 100 connections, buffer size 100, buffer service rate $\mu = 1$, average random processing time 10. The dashed waterline is the theoretical average buffer size computed from the limiting approximation (32).

5.5 Window size distribution

5.5.1 Theoretical plots

Figure 8 displays the theoretical function $g(x)$ and figure 9, the theoretical primitive of $g(x)$.

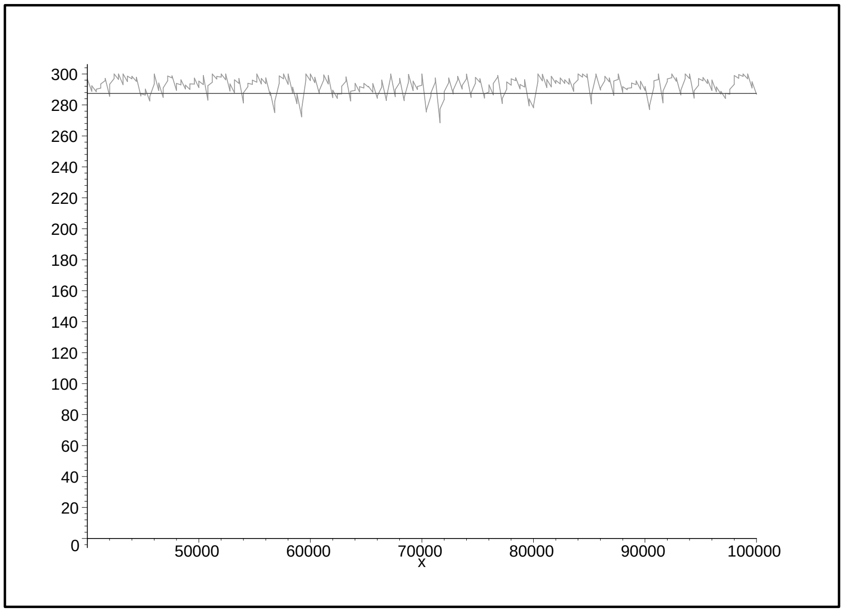


Figure 5: Histogram of buffer occupancy versus time for simplified TCP, for 100 connections, buffer size 300, buffer service rate $\mu = 1$, average random processing time 10. The dashed waterline is the theoretical average buffer size computed from the limiting approximation (32).

5.5.2 Simplified TCP

Figure 10 display the simulated window distribution. The distribution is obtained after having frozen the simulation at a certain time. Notice that non-integer values are attainable.

5.5.3 TCP Reno

Figure 11 displays the distribution of window size with TCP reno, simulated for 100 permanent connections with buffer size 800, service rate $\mu = 1$, average processing time 10.

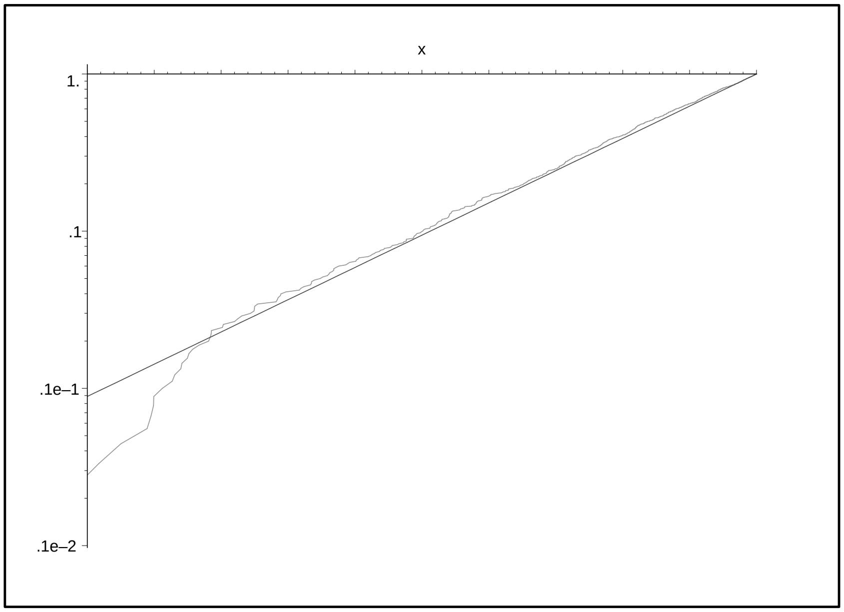


Figure 6: Repartition function of buffer occupancy, simplified TCP, for 100 connections, buffer size 100, buffer service rate $\mu = 1$, average random processing time 100. The straight line is the theoretical exponential distribution computed via the approximated formula (32).

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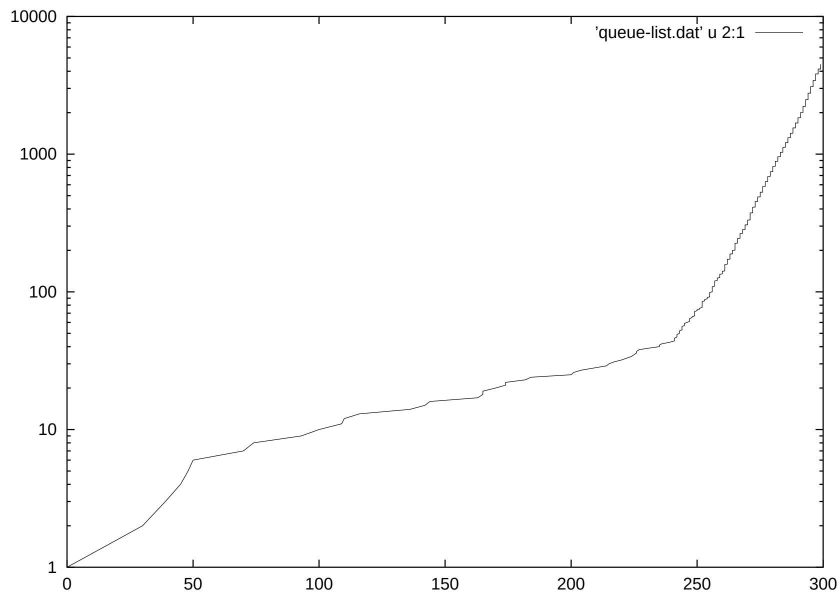

Figure 7: Repartition function of buffer occupancy, TCP Reno, for 100 connections, buffer size 300, buffer service rate $\mu = 1$, average random processing time 10.

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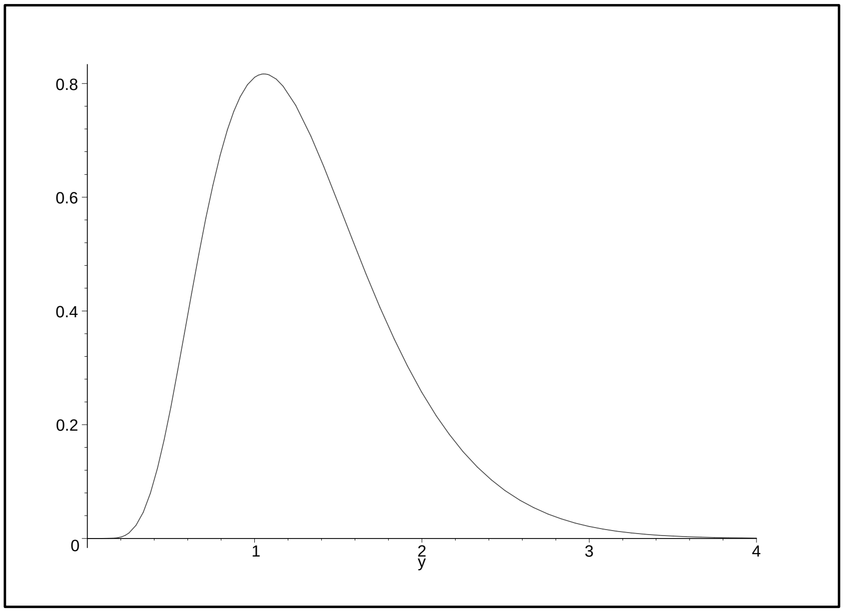

Figure 8: Theoretical function $g(x)$ for window size distribution.

Appendix

5.5.4 Discrete model

It was mentioned in section 3.2 that in discrete model the window length is supposed to be integer, and so is the length of the free buffer (to make the text more readable we write below 'buffer' instead of 'free buffer'.) The length of a window is measured in number of packets this window addresses to the buffer. It is supposed that the length w_I of I-th window is changed at a moment the window is addressed:

$$\begin{array}{ll} w_I \to w_I + 1 & \text{if the buffer length } r \geq w_I \\ w_I \to w_I/2 & \text{if the buffer length } r < w_I \end{array}$$

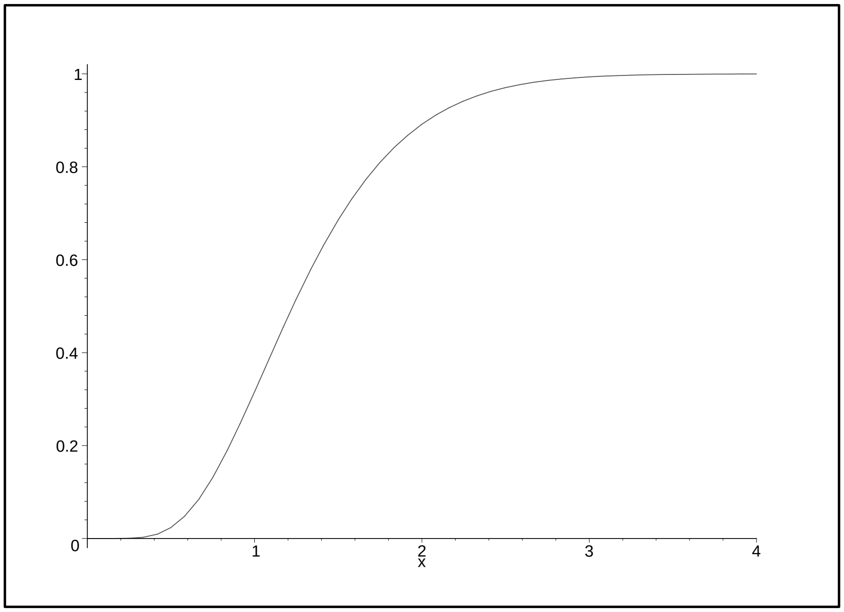

Figure 9: Primitive theoretical of function $g(x)$ for window size distribution.

The length of buffer r is measured in number of new packets it can room, this number changes at moments the windows are addressed and at moments a packet is processed and leaves the buffer,

$r \to \max\{0, r - i\}$	when a window of length i is addressed
$r \to r + 1$	when a packet is processed

The number of windows is N. Denote by $w_i^{(N)}(t)$ the fraction of windows of length i at time moment t, $i = 0, 1, ...$, $\mathbf{w} = \{w_i\}_{i=0}^{\infty}$, $\sum_i w_i = 1$. Denote by $R_j^{(N)}(t)$ the probability that at time moment t the length of buffer is at least j, $j = 0, 1, 2, ..$, $\mathbf{R}^{(N)} = \{R_j^{(N)}\}_{j=0}^{\infty}$.

The windows are addressed at time moments that form a Poisson flow of intensity λ. The windows are i.i.d distributed and are addressed randomly. A packet service time is distributed exponentially with mean μ.

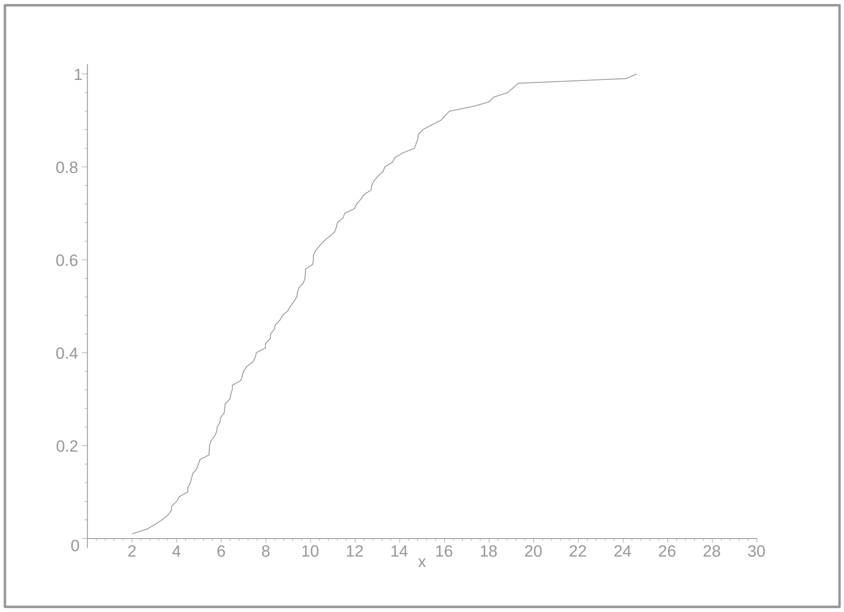

Figure 10: Window size distribution frozen at a random time for simplified TCP, for 100 connections, buffer size 800, buffer service rate $\mu = 1$, average random processing time 100.

Thus, if the buffer length at moment t is j, then at moment $t + \Delta t$ it may becomes $j + 1$ or $j - k,\ k = 1, ..., j$ (Δt is supposed to be 'very small'). The description of the model suggests that the buffer distribution function $\mathbf{R}^{(N)}(t)$ changes following an equation

$$\frac{dR_j^{(N)}(t)}{dt} = \mu(R_{j-1}^{(N)}(t) - R_j^{(N)}(t)) - \lambda \sum_{k=0}^{\infty}(R_j^{(N)}(t) - R_{j+k}^{(N)}(t))w_k^{(N)}(t) \tag{43}$$

$$= \mu(R_{j-1}^{(N)}(t) - R_j^{(N)}(t)) - \lambda R_j^{(N)}(t) + \lambda \sum_{k=0}^{\infty} R_{j+k}^{(N)}(t) w_k^{(N)}(t).$$

Further, fraction of windows $w_i^{(N)}$ at time moment t may become $w_i^{(N)} \pm \frac{1}{N}$ at moment $t + \Delta t$ if a window is addressed during time interval Δt. Thus the changes of the value $\mathbf{R}^{(N)}$

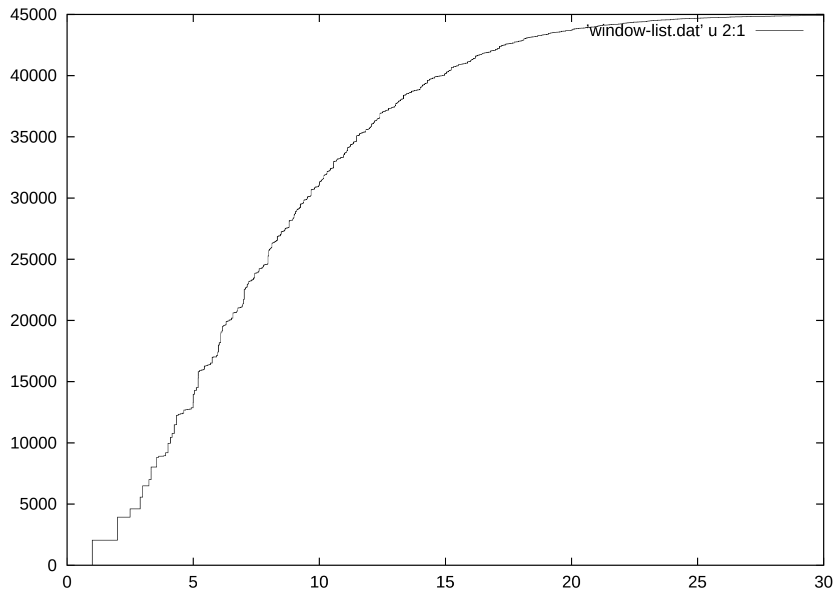

Figure 11: Window size distribution sampled at periodic time for TCP Reno, for 100 connections, buffer size 800, buffer service rate $\mu = 1$, average random processing time 10.

are $\mathrm{O}(N)$ times larger than the changes of value $\mathbf{w}^{(N)}$. Intuitively,considering $\frac{1}{N}$ as a small parameter we can describe the evolution of $\mathbf{R}^{(N)}$ using the technic adjusted for equations with slowly changing coefficients $w_i^{(N)}$.

To be more rigorous consider a Markov process $\left(\mathbf{w}^{(N)}, \mathbf{R}^{(N)}\right)$. The generating operator of the process is

$$A_N f(\mathbf{w}^{(N)}, \mathbf{R}^{(N)})(t) \tag{44}$$
$$= \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \left(f(\mathbf{w}^{(N)}(t+\Delta t), \mathbf{R}^{(N)}(t+\Delta t)) - f(\mathbf{w}^{(N)}(t), \mathbf{R}^{(N)}(t)) \right),$$

Here $f(\mathbf{w}^{(N)}, \mathbf{R}^{(N)})$ is a function defined on the sequences $\mathbf{w}^{(N)}, \mathbf{R}^{(N)}$. We suppose that $\dfrac{\partial f}{\partial w_i}, \dfrac{\partial f}{\partial R_i}, \dfrac{\partial^2 f}{\partial w_i^2}, \dfrac{\partial^2 f}{\partial R_i^2}$ are bounded. Rewrite (44) in the form

$$A_N f(\mathbf{w}^{(N)}, \mathbf{R}^{(N)})(t) \tag{45}$$
$$= \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \Big(f(\mathbf{w}^{(N)}(t+\Delta t), \mathbf{R}^{(N)}(t+\Delta t)) - f(\mathbf{w}^{(N)}(t+\Delta t), \mathbf{R}^{(N)}(t))$$
$$+ f(\mathbf{w}^{(N)}(t+\Delta t), \mathbf{R}^{(N)}(t)) - f(\mathbf{w}^{(N)}(t+\Delta t), \mathbf{R}^{(N)}(t)) \Big).$$

The first two summands of (45) can be presented in the form

$$\sum_{j=0}^{\infty} \frac{\partial f(\cdot, \mathbf{R}^{(N)})}{\partial R_j} \frac{dR_j}{dt} + \mathrm{O}\left(\frac{1}{N}\right) \tag{46}$$

$$= \sum_{j=0}^{\infty} \frac{\partial f(\cdot, \mathbf{R}^{(N)})}{\partial R_j} \left(\mu(R_{j-1}^{(N)} - R_j^{(N)}) - \lambda R_j^{(N)} + \lambda \sum_{k=0}^{\infty} R_{j+k}^{(N)} w_k^{(N)} \right) + \mathrm{O}\left(\frac{1}{N}\right).$$

Two last summands of (45) can be presented in the form

$$\lambda \sum_{i=0}^{\infty} \left(f(\mathbf{w}^{(N)} - \frac{e_i}{N}, \cdot) - f(\mathbf{w}^{(N)}, \cdot) \right) w_i \tag{47}$$

$$+ \left(f(\mathbf{w}^{(N)} + \frac{e_i}{N}, \cdot) - f(\mathbf{w}^{(N)}, \cdot) \right) (w_{i-1}^{(N)} R_{i-1}^{(N)} + w_{2i}^{(N)}(1 - R_{2i}^{(N)}) + w_{2i-1}^{(N)}(1 - R_{2i-1}^{(N)}))$$

$$= \frac{\lambda}{N} \sum_{i=0}^{\infty} \frac{\partial f(\mathbf{w}^{(N)}, \cdot)}{\partial w_i} \left(w_{i-1}^{(N)} R_{i-1}^{(N)} - w_i + w_{2i}^{(N)}(1 - R_{2i}^{(N)}) + w_{2i-1}^{(N)}(1 - R_{2i-1}^{(N)}) \right)$$

$$+ \mathrm{O}\left(\frac{\partial^2 f}{\partial w_i^2} \frac{1}{N^2} \right).$$

Here e_i is a vector $e_i = (0, ..., 0, 1, 0, ...)$ with 1 at the ith place.

Consider a dynamic system defined on the real value sequences $\mathbf{w} = \{w_i\}_{i=0}^{\infty}$, $\mathbf{R} = \{R_j\}_{j=0}^{\infty}$:

$$\frac{dR_j(t)}{dt} = \mu(R_{j-1}(t) - R_j(t)) - \lambda R_j(t) + \lambda \sum_{k=0}^{\infty} R_{j+k}(t) w_k(t), \tag{48}$$

$$\frac{dw_i(t)}{dt} = \frac{\lambda}{N} \left(w_{i-1}(t) R_{i-1}(t) - w_i(t) + w_{2i}(t)(1 - R_{2i}(t)) + w_{2i-1}(t)(1 - R_{2i-1}(t)) \right), \tag{49}$$

$$\sum_i w_i = 1, \quad R_j \geq R_{j+1}, \quad R_0 = 1, \quad R_j \to 0 \text{ as } j \to \infty, \tag{50}$$

with given $\mathbf{w}(0)$, $\mathbf{R}(0)$.

System (48) – (50) is a system with small parameter $\frac{1}{N}$.

Our aim is to investigate the behavior of solutions $(\mathbf{R}, \mathbf{w})$ to (48) – (50) as $N \to \infty$ and to show that as $N \to \infty$ Markov process described above converges to the dynamic system defined by (48) – (49).

First, the properties of solution $(\mathbf{R}, \mathbf{w})$ have to be investigated. For example, it has to be show that R_i, w_i decrease sufficiently fast as $i \to \infty$. Also, introducing the metric $\rho((\mathbf{w}, \mathbf{R}), (\mathbf{w}', \mathbf{R}') = \sup_i \{|w_i - w_i'|/(i+1), |R_i - R_i'|/(i+1)\}$ on a set of sequences $(\mathbf{w}, \mathbf{R})$

one has to shown that a solution to linearized equations $(48),(49)$ continuously depends on initial data for any finite $t \leq T < \infty$.

The convergence of Markov process to the dynamic system means that for a space of smooth functions f defined on the sequences $(\mathbf{w}, \mathbf{R})$

$$\lim_{N\to\infty} \mathbf{E}|f(\mathbf{w}^{(N)}(t), \mathbf{R}^{(N)}(t)) - f(\mathbf{w}(t), \mathbf{R}(t))| = 0 \tag{51}$$

if $\mathbf{w}^{(N)}(0) \to \mathbf{w}(0), \quad \mathbf{R}^{(N)}(0) \to \mathbf{R}(0)$ as $N \to \infty$.

Equality (51) follows (see [EK]) from the fact that on a properly chosen space of functions f the generator $A_N f(\mathbf{w}^{(N)}, \mathbf{R}^{(N)})$ tens to $\lim_{N\to\infty} A_{(N)}(\mathbf{w}, \mathbf{R}) = \dfrac{df(\mathbf{R}, \mathbf{w})}{dt}$, where $(\mathbf{R}, \mathbf{w})$ is a solution to $(48) - (49)$. We will not present the proof of convergence here, one can find a similar proof, for example, in [6].

Let us consider now equations $(48),(49)$ and an equation of type (48) but with coefficients $\tilde{w}_k$ not depending on t, say $\tilde{w}_k = w_k(0)$:

$$\frac{d\tilde{R}_j(t)}{dt} = \mu(\tilde{R}_{j-1}(t) - \tilde{R}_j(t)) - \lambda \tilde{R}_j(t) + \lambda \sum_{k=0}^{\infty} \tilde{R}_{j+k}(t)\tilde{w}_k, \tag{52}$$

Equation (52) has a stationary solution

$$R_j^{(loc)} = e^{-a_0 j}, \tag{53}$$

where a_0 is a solution to equality

$$e^a - 1 = \frac{\lambda}{\mu}(1 - \sum_{k=1}^{\infty} e^{-ak}\tilde{w}_k). \tag{54}$$

If a_0 is close to 0, then (54) gives

$$a_0 = \frac{\lambda}{\mu}(1 - \sum_{k=1}^{\infty} e^{-a_0 k}\tilde{w}_k).$$

It is easy to show that for any initial values $\tilde{R}_j(0)$

$$\tilde{R}_j(t) \to R_j^{(loc)} \quad \text{as} \quad t \to \infty.$$

Let $\Delta_j(t) = R_j^{(loc)} - \tilde{R}_j(t)$. If $\Delta_j(0) \geq 0$ (≤ 0), then $\Delta_j(t) \geq 0$ (≤ 0). Really, it is sufficient to suppose, that $\Delta_j(0) > 0$ (or $\Delta_j(0) < 0$) as $t < t_0$ and $\Delta_j(t_0) = 0, \quad t_0 \geq 0$. Than, by (52), $\frac{\Delta_j(t_0)}{dt} > 0$ $(\frac{\Delta_j(t_0)}{dt} < 0)$, thus $\Delta_j(t) \geq 0$ (≤ 0) $\forall t > 0$.

Further, for $\Delta_j > 0$ (< 0) consider $\sum_j |\Delta_j(t)|$. By (52)

$$\frac{d\sum_{j=0}^{\infty}|\Delta_j(t)|}{dt} = -\mu|\Delta_1(t)| - \lambda\sum_{j=0}^{\infty}|\Delta_j(t)|(1-\sum_{k=0}^{j}\tilde{w}_k) < 0.$$

Integrating over t we get that $\int_0^\infty |\Delta_1(t)|dt < \infty$, and $\Delta_1(t) \to 0$. Using induction in k one shows that $\forall\ k \quad \Delta_k(t) \to 0$. For $R_j^-(0) = \min\{R_j^{(loc)}, R_j(0)\}$, $R_j^+(0) = \max\{R_j^{(loc)}, R_j(0)\}$ we have $R_j^-(t) \le \tilde{R}_j(t) \le R_j^+(t)$, $|R_j^\pm(t) - R_j^{(loc)} \to 0$ as $t \to \infty$, thus $\tilde{R}_j(t) \to R_j^{(loc)}$ as $t \to \infty$. To estimate the rate of convergence consider $\tilde{R}_j(0) = e^{-aj}$. By (52)

$$\frac{da}{dt} = \mu(e^a - 1) - \lambda(1 - \sum_{k=1}^{\infty} e^{-ak}\tilde{w}_k). \tag{55}$$

Equations show that $\frac{da}{dt} < 0$ if $a > a_0$ and $\frac{da}{dt} > 0$ if $a < a_0$ and the rate of convergence depends on $|a - a_0|$ and on $\tilde{\mathbf{w}}$. That means that for any $\varepsilon > 0\ \exists T$, $|\tilde{R}_j(t) - R_j^{loc}| < \varepsilon$ as $t \ge T$.

Let us turn to equation (49). Here it is natural to rescale the time variable, $tN \to t_1$.

Remark that the coefficients $R_j(t)$ are close to $\tilde{R}_j(t)$ for $t \ge T,\ t_1 \ge T_1/N$ and $w_i(t)$ are close to $w_i(0)$ for $0 < t_1 \le 2T_1 = 2T/N$ if N is sufficiently large. Therefore for $T_1 \le t_1 \le 2T_1$ solution to (7) is close to solution of equation

$$\frac{dw_i(t)}{dt} = \lambda\left(w_{i-1}R_{i-1}^{(loc)} - w_i + w_{2i}(1 - R_{2i}^{(loc)}) + w_{2i-1}(1 - R_{2i-1}^{(loc)}\right),$$

with initial data $\mathbf{w}(0)$.

Let $\tilde{\mathbf{R}}$ be a solution to equation of type (52)

$$\frac{d\tilde{R}_j(t)}{dt} = \mu(\tilde{R}_{j-1}(t) - \tilde{R}_j(t)) - \lambda\tilde{R}_j(t) + \lambda\sum_{k=0}^{\infty}\tilde{R}_{j+k}(t)\tilde{w}_k(t), \tag{56}$$

where $\tilde{w}_k(t) = w_k(lT)$ as $lT \le t < (l+1)T$, $l = 2, 3, ...$ During time intervals of length $T = T_1/N$ the values of $\mathbf{w}$ change little if N is sufficiently large, therefore the values of $\tilde{\mathbf{R}}(t)$ stay in ε_1 neighborhood of $\mathbf{R_l}^{(loc)}$, $l = 1, 2, ...,$ $\varepsilon_1 = \mathrm{O}(\varepsilon)$, where $\mathbf{R_l}^{(loc)} = \mathbf{R_l}^{(loc)}(\mathbf{w}(lT_1))$, $(R_l^{(loc)})_j = e^{-ja_l}$,

$$e^{a_l} - 1 = \frac{\lambda}{\mu}(1 - \sum_{k=1}^{\infty} e^{-a_l k}w_k(lT_1).$$

The values of solution to (49) depend continuously on coefficient. Therefore for any $t < \infty$ there exists such sufficiently large N_0 that for $N > N_0$ the values of solution to (49) are close to the solution to (56). And as $N \to \infty$ for any $t < \infty$ solution to (56) tens to solution of equation

$$\frac{dw_i(t)}{dt} = \lambda \left(w_{i-1} R^{(loc)}_{i-1}(\mathbf{w}(t) + w_{2i}(1 - R^{(loc)}_{2i}(\mathbf{w}(t)) + w_{2i-1}(1 - R^{loc}_{2i-1}(\mathbf{w}(t)\right), \tag{57}$$

where $\mathbf{R}^{(loc)}(\mathbf{w}(t))$ is defined by (53) with $\mathbf{w} = \mathbf{w}(t)$.

5.5.5 Continuous model

If the number of packets in a window increases, the intensity λ of the moments the windows are addressed increases and the service time of a packet decreases we come to an equation for $\mathbf{w}(t)$

$$\frac{dw_i(t)}{dt} = \frac{\lambda}{N} \left(w_{i-H}(t) R_{i-H}(t) - w_i(t) + w_{2i}(t)(1 - R_{2i}(t)) + w_{2i-1}(t)(1 - R_{2i-1}(t)) \right), \tag{58}$$

where H is a large number. Solution ($\mathbf{R}, \mathbf{w}$ to (48), (50),(58)has the properties similar to the properties of solution to (48)-(50).

Next step is to consider a continuous model, for which (58) is a difference approximation.

Here the window length is supposed to be a real value, and so is the length of the buffer. It is supposed that the length w_I of I-th window is changed at a moment the window is addressed:

$w_I \to w_I + 1$	if the buffer length	$r \geq w_I$
$w_I \to w_I/2$	if the buffer length	$r < w_I$

The length r of the buffer changes at moments the windows are addressed and as some data is processed and leaves the buffer,

$r \to \max\{0, r - y\}$	when a window of length y is addressed
r increases with rate	μ as some data is processed

The number of windows is N. Let $w^{(N)}(y)$ be the probability distribution of window lengths and let $R^{(N)}(x)$ be the probability that the buffer length is at least x. As $N \to \infty$ the performance of Markov process converges to a dynamic system defined by the equations

$$\frac{\partial R(x,t)}{dt} = \mu \frac{\partial R(x,t)}{\partial x} - \lambda R(x,t) + +\lambda \int_{k=0}^{\infty} R(x+y,t) w(y,t) dt, \tag{59}$$

$$\frac{\partial w(y,t)}{\partial t} = \frac{\lambda}{N} \left(w(y-1,t) R(y-1,t) - w(y,t) + 2w_(2y,t)(1 - R(2y,t)) \right), \tag{60}$$

$$\int w(y) dy = 1, \quad R(x,t) \text{ decreases in } x, \quad R_0 = 1, \quad R(x,t) \to 0 \text{ as } x \to \infty, \tag{61}$$

with given $\mathbf{w}(0)$, $\mathbf{R}(0)$.

To prove that as the number of windows $N \to \infty$ the solution to (59)−)61) represents the limit of a Markov process one has to consider a smooth functionals $f(w^{(N)}(x,t), R^{(N)}(x,t)$ and $f(w(x,t),(x,t)$ where (w,R) is a week solution to (59)−(61) and to show the convergence of the first functional to the second one. Such technic is used in [7], we will not deal with it here.

For given $w = \tilde{w}(y)$ the stationary solution to (59) has the form

$$R^{(loc)}(x) = e^{-ax}, \tag{62}$$

where a selves the equality

$$a = \frac{\lambda}{\mu}(1 - \int_0^\infty e^{-as} w(s) ds). \tag{63}$$

As $N \to \infty$ the solution to (59) − (61) converges to solution of equation

$$\frac{\partial w(y,t_1)}{\partial t_1} = \lambda \left(w(y-1,t) R^{(loc)}(y-1,t) - w(y,t) + 2w_(2y,t)(1 - R^{(loc)}(2y,t)) \right), \tag{64}$$

$$R^{(loc)}(y,t) = e^{-a(t)y}, \quad a(t) = \frac{\lambda}{\mu}(1 - \int_0^\infty e^{-as} w(s,t) ds).$$

The proof of this convergence repeats the proof of the discrete case.

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